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A STANDARD FOR THE SMEAR

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ABSTRACT

We present a notation for "smear" that makes its definition clear and unambiguous. Instead of proposing a unique definition, we define and recommend the concept of the "smear of a function," $S(f)$, and allow individual researchers to freely choose the function f. The notation is clear enough that it allows to easily compare results for the smear of different functions. We also recommend that, for more detailed comparisons, authors also compute the five basic statistical quantities of the distribution. We collect all definitions of smear presently in use, expressed in our notation, and compare them. Finally, we present a new definition which we suggest be used as a standard of comparison among different calculations and/or experiments.

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1. Introduction

The advent of large circular accelerators[1,2] has brought forward a new consideration in their design: because of their large size, economic reasons force the use of superconducting magnets of small bore. By their nature, these magnets have relatively large magnetic field nonlinearities which cause nonlinearities in the dynamics. Ideally, the motion of the particles should be as linear as possible in order to ensure the predictability of their behavior and therefore the operational reliability of the machine. The larger the size of the bore, the more linear the particle dynamics, but the more expensive the machine. Too small a bore, and the machine is unreliable. Therefore, economic reasons impose the necessity to tolerate a certain amount of nonlinearity in the particle motion.

One way to quantify the deviation from linearity in the particle motion is to measure or calculate the linear invariants turn by turn. The deviation from constancy therefore provides a measure of the nonlinearity. A quantity that measures the size of this deviation[3] has been called the "smear." At present, however, there are so many definitions of smear in use that comparisons between different experiments and/or calculations has become quite confusing. In this note we present and recommend a notation that allows a clear definition. Our recommendation is to adopt the concept of the "smear of a function," $S(f)$, defined by Eq. (2.5), rather than to try to define smear itself. In the cases of interest to us, namely experiments and simulations[4–6] in circular accelerators, f is usually a function of the amplitudes or the Courant–Snyder invariants. By appropriately choosing f, we reproduce all "rms–type" smear definitions in use so far. It is up to each individual to spell out which f she or he is using. While researchers cannot be forced to use the same definition of smear, at least our notation is flexible and clear enough that it makes it possible to compare different calculations and experimental results with relative ease.

2. Basic definitions

Consider the cluster of points in Fig. 1, described by the generic coordinates u and v. A qualitative definition of the smear of this distribution is: "size of the cluster divided by its distance from the origin." In the cases of practical interest to us, the points represent the turn-by-turn measurements of the horizontal and vertical amplitudes or the Courant–Snyder invariants. If the motion were perfectly linear and uncoupled, the distribution would reduce to a point, yielding zero smear. If the motion were very nonlinear, the points would be widely distributed, the smear would be large and its very concept useless as a measure of first order departure from linearity. Thus we are interested only in reasonably clustered distributions in the first quadrant.

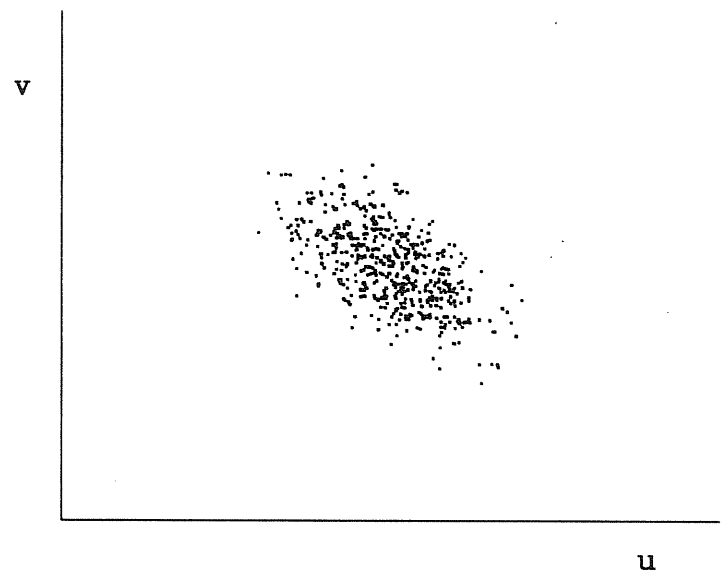

Fig. 1. A sample distribution of points with generic coordinates u and v.

The many definitions of smear in use today have the same qualitative meaning stated above. For example, the generic coordinates u and v might be the horizontal and vertical particle amplitudes a_x and a_y measured at some specific point in the lattice;[4] or they might be the Courant–Snyder invariants $2J_x$ and $2J_y$ (or the eigen-invariants $2J_1$ and $2J_2$, if there is linear coupling),[6–11] or they might be linear combinations of the invariants that emphasize or deemphasize specific coupling resonances,[5] or that emphasize, say, the horizontal size over the vertical size. In addition, one might choose either "rms" quantities or "max–min" quantities[8,9,12] to define the size of the distribution. Here we shall focus on rms quantities only and will collect the max–min definitions of smear in use in order to recommend a specific notation for each one of them.

A distribution of points is fully specified by an infinite set of moments; in practice, of course, we want to measure or calculate only one or a few relevant quantities, depending on the purpose for which they are intended. For example, comparison between tracking codes, or between simulation and experiment, may require a more detailed description of the distribution than, say, a specification of needed magnet quality. In all cases of interest so far, and in the foreseeable future, the specification of the first five moments of the distribution is quite sufficient. These are the two averages

$$\bar{u} = \langle u \rangle , \qquad \bar{v} = \langle v \rangle \tag{2.1}$$

and the three covariances

$$\chi(q,p) = \langle (q-\bar{q})(p-\bar{p}) \rangle = \langle qp \rangle - \langle q \rangle \langle p \rangle \tag{2.2}$$

where $q, p = u$ or v. The average $\langle f(u,v) \rangle$ of an arbitrary function $f(u,v)$ is defined by

$$\overline{f(u,v)} \equiv \langle f(u,v) \rangle = \frac{1}{N} \sum_{t=1}^{N} f(u_t, v_t) \tag{2.3}$$

which is a turn-by-turn average in physical applications.

The usual σ's and the correlation C are

$$\sigma_u = \sqrt{\chi(u,u)}, \qquad \sigma_v = \sqrt{\chi(v,v)}, \qquad C = \chi(u,v) \tag{2.4}$$

Note that the correlation can be positive or negative depending on the overall orientation of the cluster. Roughly speaking, $C > 0$ if it lies along the main diagonal, $C < 0$ if it lies along the second diagonal (as is the case in Fig. 1). In general, C is in the range $-\sigma_u \sigma_v \le C \le \sigma_u \sigma_v$.

We define the smear of the function f, $S(f)$, by

$$S(f)^2 \equiv \frac{\chi(f,f)}{\langle f \rangle^2} \tag{2.5}$$

which is nothing but the normalized rms of the function.

3. Properties of the smear of a function

A nice property of $S(f)$ is scale invariance, *i.e.*,

$$S(\lambda f) = S(f) \tag{3.1}$$

where λ is a constant; this means that $S(f)$ is not sensitive to the absolute size of the distribution.

The smear of a linear function is given in terms of the 5 moments mentioned above. Thus, for $f(u,v) = au + bv + c$, we obtain

$$S(au + bv + c) = \frac{\sqrt{a^2\sigma_u^2 + b^2\sigma_v^2 + 2abC}}{|a\bar{u} + b\bar{v} + c|} \tag{3.2}$$

The function f can also be a *vector function* of u, v, if we adopt the obvious generalization of the definition of the covariances to be

$$\chi(\mathbf{f}, \mathbf{g}) \equiv \left\langle (\mathbf{f} - \bar{\mathbf{f}}) \cdot (\mathbf{g} - \bar{\mathbf{g}}) \right\rangle \tag{3.3}$$

For example, for $\mathbf{r} = (u, v)$, we obtain

$$S(\mathbf{r}) = \frac{\sqrt{\chi(\mathbf{r}, \mathbf{r})}}{|\langle \mathbf{r} \rangle|} \equiv \frac{\sigma_{\mathbf{r}}}{|\langle \mathbf{r} \rangle|} = \sqrt{\frac{\sigma_u^2 + \sigma_v^2}{\bar{u}^2 + \bar{v}^2}} \tag{3.4}$$

This latter definition of the smear, $S(\mathbf{r})$, is perhaps the one in most straightforward correspondence with the qualitative definition stated in the Introduction. The numerator is an obvious measure of the size of the distribution and the denominator an obvious measure of its distance from the origin; it is insensitive to the orientation and shape of the distribution.

For small–smear distributions it is easy to prove that

$$\begin{aligned} \langle q^2 \rangle &= \langle q \rangle^2 \times [1 + \mathcal{O}(S^2)] \\ \chi(q^2, p^2) &= 4\bar{q}\bar{p}\,\chi(q, p) \times [1 + \mathcal{O}(S)] \end{aligned} \tag{3.5}$$

where, again, q and p can be either u or v. Thus one can approximately relate the five moments and the smear of amplitude distributions to Courant–Snyder distributions, as we state more explicitly below.

Whatever its definition, the smear must satisfy the important property of being a first order invariant, *i.e.*, it must be invariant under translation through linear elements along the lattice. This requirement ensures that the smear is approximately independent of the observation point (random nonlinearities cause relative deviations of $\mathcal{O}(S)$ when the smear is measured at different lattice points; however, nonlinear magnetic elements, such as chromaticity sextupoles, can cause large relative deviations in the smear in certain regions of the lattice, if the phase difference between them is appropriate). All definitions presented in the next section, except S_1, are linear invariants.

4. Collection of smear definitions

In Table 1 below we present a collection of definitions of smear, expressed with our notation. The Courant–Snyder invariants $2J_x$, $2J_y$ are related to the lattice functions by $2J_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2$, and similarly for $2J_y$. The amplitudes a_x, a_y, which have the advantage of being directly measurable, are related to the invariants by $a_i = \sqrt{2\beta_i J_i}$ (we assume, for simplicity, that they are measured at a point where $\alpha_x = \alpha_y = 0$). If the motion has linear horizontal–vertical coupling, the eigen–invariants $2J_1$, $2J_2$ should normally be used in place of $2J_x$, $2J_y$; otherwise, the smear does not vanish at zero amplitude.[5] If f is only a function of J_x, J_y (or J_1, J_2), the linear–invariance property is guaranteed. If f is a function of the amplitudes a_x, a_y, some care is needed. Clearly $S(a_x)$ and $S(a_y)$ are linear invariants since the β–function cancels out on account of the scale invariance propety; however, $S(a_x + a_y)$ and $S(\mathbf{a})$ are not.

Table 1. RMS smear definitions.

Name	Definition	Comments
S_1	$S(\mathbf{a})$	Ref. [3]
S_2	$S(a_x)$	Ref. [4]
S_3	$\max[S(a_x), S(a_y)]$	Ref. [7]
S_4	$1.56 \max[S(a_x), S(a_y)]$	Ref. [9]
S_5	$\max[S(J_x), S(J_y)]$	
S_6	$S(J_x + J_y)$	Ref. [5]
S_7	$S(J_1 + J_2)$	Ref. [6]
S_8	$S(\mathbf{J})$	new

Actually Ref. [3] contains only a descriptive definition of smear, without a formula; the definition $S(\mathbf{a})$ in the table above corresponds to Alex Chao's best recollection of what was then referred to as smear.

We present now the max–min definitions of smear that are in use. In these formulas $A = \sqrt{2J}$, and *not* $\sqrt{2\beta J}$ (of course, if $\beta_{x,\max} = \beta_{y,\max}$ one can take A to be $\sqrt{2\beta_{\max} J}$). $\hat{A}$ and $\check{A}$ are the maximum and minimum of the distribution, and the averages are defined by $\bar{A} = (\hat{A} + \check{A})/2$, and not according to Eq. (2.3).

$$\begin{aligned} S_T &= \frac{\sqrt{2}}{3} \frac{\max[\hat{A}_x - \check{A}_x, \hat{A}_y - \check{A}_y]}{\sqrt{\bar{A}_x^{\,2} + \bar{A}_y^{\,2}}} && \text{Ref. [12]} \\ S_T' &= \frac{1}{3} \max\left[\frac{\hat{A}_x - \check{A}_x}{\bar{A}_x}, \frac{\hat{A}_y - \check{A}_y}{\bar{A}_y}\right] && \text{Ref. [8]} \end{aligned} \tag{4.1}$$

The factor $\frac{1}{3}$ in front of S_T' is purely *ad hoc*; it was included in its definition as a rough guess to make it agree with S_3 in practical applications to the SSC* (however, read on).

The "10% criterion" for the linear aperture[3] of the SSC was first stated in print with an explicit formula, to the best of our knowledge, in terms of S_T' in Ref. [8], where the linear aperture was defined by $S_T' = 0.10$. It was then

⋆ Alex Chao, public confession.

observed empirically, in analytical and tracking studies on a specific SSC lattice,[7] that $S'_T = 0.10$ corresponds to $S_3 \approx 0.064$. This provided the motivation in Ref. [9] (where S_4 was called $S_{\rm rms}$) for the factor 1.56 in front of S_4, since $1.56\times 0.064 = 0.10$. Thus the linear aperture criterion can be stated as $S'_T = 0.10$, or $S_3 = 0.064$, or $S_4 = 0.10$, or $S_5 = 0.128$ (see below for the explanation of $S_5 \approx 2S_3$).

It should be remarked that the smear is not a statistical quantity; that is to say, once the set of all nonlinearities for a given machine is fully specified, the smear, in either of its rms or max–min versions can, in principle, be calculated analytically to any desired degree of accuracy. In tracking simulations the computation of the smear takes on an apparently statistical character on account of the finite number of turns used to "measure" it (experience shows, however, that simulations with as few as 500–1,000 turns yield very accurate values). In this sense the "rms" qualification can be misleading. In model machines with random errors the smear is truly a statistical quantity, since it requires an ensemble average over an infinite number of machines on top of whatever other calculation has to be performed for a sample element of the ensemble. For these types of calculations the rms versions of the smear are the appropriate ones.

5. Comparison among the different definitions

For small–smear distributions one obtains, from Eq. (3.5),

$$\begin{aligned} \langle 2J_i \rangle &\approx \langle a_i \rangle^2 / \beta_i \\ \chi(2J_i, 2J_j) &\approx 4\bar{a}_i \bar{a}_j \chi(a_i, a_j)/(\beta_i \beta_j) \end{aligned} \tag{5.1}$$

for $i, j = x$ or y. It follows that $S(J_i) \approx 2S(a_i)$, and therefore $S_5 \approx 2S_3$.

For one–dimensional motion there are several relations among the different definitions. In this case, $S_1 = S_2 = S_3 = S_4/1.56$, $S_5 = S_6 = S_7 = S_8 \approx 2S_3$, and $S_T = \sqrt{2}\, S'_T$.

For two–dimensional motion there is no direct quantitative comparison among the several definitions (except $S_5 \approx 2S_3$). In particular, the SPS definitions S_6 and S_7 are constructed so that they are sensitive to the orientation of the cluster of points, yielding small values when there is coupling between the horizontal and vertical planes of motion, or between the eigenplanes, respectively. Thus they respond in a qualitatively different way to coupling resonances than the other definitions, all of which respond similarly. As we mentioned above, the relation $S'_T \approx 1.56\, S_3$ was found empirically for an SSC lattice; it is not meant to be generally valid, especially under the influence of significant resonances.

A numerical comparison among the different definitions, for specific lattice examples, will be provided elsewhere[13] once this craziness of the move to Texas is over.

6. Conclusions and recommendations

In conclusion, we recommend that authors:

1. adopt the definition of the smear of a function, Eq. (2.5), and that they state explicitly their choice of function f, which should be a linear invariant; and

2. provide the 5 basic moments of the quantities that appear in f; for example, users of S_5 ought to state the values of $\langle J_x \rangle$, $\langle J_y \rangle$, $\chi(J_x, J_x)$, $\chi(J_y, J_y)$ and $\chi(J_x, J_y)$.

We also suggest that the definition of smear $S_8 \equiv S(\mathbf{J})$ defined by Eq. (3.4) with $\mathbf{r} = (J_x, J_y)$ (or $\mathbf{r} = (J_1, J_2)$ if there is linear coupling), be adopted for purposes of comparison among different calculations and/or experiments. This "standard" smear is insensitive to the shape and orientation of the cluster of points, and is sensitive only to its overall size. Also, its square is directly amenable to theoretical calculation.[7,10,11]

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